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LETTER TO THE EDITOR

Integrable boundaries, conformal boundary conditions and A-D-E fusion rules

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Abstract. The $sl(2)$ minimal theories are classified by a Lie algebra pair (A, G) where G is of A-D-E type. For these theories on a cylinder we propose a complete set of conformal boundary conditions labelled by the nodes of the tensor product graph $A \otimes G$. The cylinder partition functions are given by fusion rules arising from the graph fusion algebra of $A \otimes G$. We further conjecture that, for each conformal boundary condition, an integrable boundary condition exists as a solution of the boundary Yang–Baxter equation for the associated lattice model. The theory is illustrated using the (A_4, D_4) or three-state Potts model.

1. Introduction

The study of conformal boundary conditions [1] continues to be an active area of research. The many areas of application include open string theory and conformal field theory, boundary critical behaviour in statistical mechanics, massive and massless boundary flows in 2D field theories as well as quantum impurities and the Kondo problem in condensed matter physics.

Conformal invariance can only exist in the presence of a boundary if the boundary and boundary conditions are invariant under local conformal transformations. This places severe restrictions on the boundary conditions in order for them to be conformal. The problem of a general classification of conformal boundary conditions has recently seen a revival of interest. For theories with a diagonal torus partition function it is known that there is a conformal boundary condition associated to each operator appearing in the theory. Moreover, the fusion rules of these boundary operators are just given by the bulk fusion algebra and thus by the Verlinde formula [2]. In contrast, for non-diagonal theories, the fusion rules are not known in general and it is not even known what constitutes a complete set of conformal boundary conditions. Indeed, these questions have only been resolved [3, 4] very recently for the simplest non-diagonal theory, namely, the critical three-state Potts model. In this letter we propose a complete set of conformal boundary conditions, fusion rules and cylinder partition functions for the $sl(2)$ minimal models.

The $sl(2)$ minimal models in the bulk are classified [5] by a pair of simply laced Dynkin diagrams (A, G) of type

$$(A,G)=\begin{cases}(A_{h-1},A_{g-1}) & \\ (A_{h-1},D_{(g+2)/2}) & g \text{ even}\\ (A_{h-1},E_6) & g=12\\ (A_{h-1},E_7) & g=18\\ (A_{h-1},E_8) & g=30.\end{cases} \tag{1}$$

Here h and g are the coprime Coxeter numbers of A and G and the central charges are

$$c = 1-\frac{6(h-g)^2}{hg}. \tag{2}$$

We conjecture that for these theories a complete set of conformal boundary conditions i and the corresponding boundary operators $\hat{\varphi}_i$ are labelled by $i \in (A, G)$:

$$\hat{\varphi}_i: \quad i=(r,a)\in(A,G) \tag{3}$$

where r, a are nodes on the Dynkin diagram of A and G respectively. We will use G to denote the Dynkin diagram and the adjacency matrix of this graph. We use r, r_1, r_2 to denote nodes of A_{h-1}; s, s_1, s_2 for the nodes of A_{g-1}; a, a_1, a_2, b for the nodes of G and i, j to label nodes in the pair (A, G).

We now introduce fused adjacency matrices (intertwiners) and graph fusion matrices. The fused adjacency matrices V_s with $s = 1, \ldots, g-1$ are defined recursively by the $sl(2)$ fusion algebra

$$V_s = V_2V_{s-1} - V_{s-2} \tag{4}$$

subject to the initial conditions $V_1 = I$ and $V_2 = G$. The matrices V_s are symmetric and mutually commuting with entries given by a Verlinde-type formula

$$V_{sa}{}^b = (V_s)_a{}^b = \sum_{m\in\mathrm{Exp}(G)} \frac{\tilde{S}_{sm}}{\tilde{S}_{1m}}\ \Psi_{am}\Psi^*_{bm} \tag{5}$$

where the columns of the unitary matrices $\tilde{S}$ and Ψ are the eigenvectors of the adjacency matrices A_{g-1} and G, respectively, and the sum is over the Coxeter exponents of G with multiplicities. We assume the graph G has a distinguished endpoint node labelled $a = 1$ such that $\Psi_{1m} > 0$ for all m. This is at least the case for A-D-E graphs. In this notation we define the fundamental intertwiner as $\hat{V}_s{}^a = V_{s1}{}^a$.

The graph fusion matrices $\hat{N}_a$ with $a \in G$ were introduced by Pasquier [6]. These are defined by the Verlinde-type formula [7]

$$\hat{N}_{ab}{}^c = (\hat{N}_a)_b{}^c = \sum_{m\in\mathrm{Exp}(G)} \frac{\Psi_{am}\Psi_{bm}\Psi^*_{cm}}{\Psi_{1m}} \qquad a,b,c\in G. \tag{6}$$

These matrices satisfy the matrix recursion relation

$$G\hat{N}_a = \sum_{b\in G} G_a{}^b \hat{N}_b \tag{7}$$

and initial conditions $\hat{N}_1 = I$ and $\hat{N}_2 = G$ where 2 denotes the unique node adjacent to 1. The numbers $\hat{N}_{ab}{}^c$ are the structure constants of the graph fusion algebra

$$\hat{N}_a\hat{N}_b = \sum_{c\in G} \hat{N}_{ab}{}^c \hat{N}_c. \tag{8}$$

All the entries of the fused adjacency matrices V_s are non-negative integers. For a proper choice of the eigenvectors and of the node 1, the entries of the graph fusion matrices $\hat{N}_a$

are also integers, and with the exception of D_{2n+1} and E_7, they are non-negative. A key identity relating the fused adjacency matrices and graph fusion matrices is

$$V_s \hat{N}_a = \sum_{b\in G} V_{sa}{}^b \hat{N}_b. \tag{9}$$

2. Fusion rules

Let i_1, i_2 and $i_3 \in (A, G)$ and consider the tensor product graph $A \otimes G$ with distinguished node $i = 1$ given by $i = (r, a) = (1, 1)$. Then we conjecture that the fusion rules for the boundary operators (3) are

$$\hat{\varphi}_{i_1} \times \hat{\varphi}_{i_2} = \sum_{i_3\in(A,G)} \mathcal{N}_{i_1 i_2}{}^{i_3} \hat{\varphi}_{i_3} \tag{10}$$

where $\mathcal{N}_{i_1}$ are just the graph fusion matrices associated with the tensor product graph $A \otimes G$

$$\mathcal{N}_{i_1 i_2}{}^{i_3} = \mathcal{N}_{(r_1,a_1)(r_2,a_2)}{}^{(r_3,a_3)} = N_{r_1 r_2}{}^{r_3} \hat{N}_{a_1 a_2}{}^{a_3} \tag{11}$$

where N_{r_1} are the graph fusion matrices for A_{h-1}. Let $\varphi_{r,s}$ be the primary chiral fields with respect to the Virasoro algebra. Then the operators $\hat{\varphi}_i = \hat{\varphi}_{r,a}$ are related to $\varphi_{r,s}$ by the intertwining relation

$$\sum_{b\in G} \hat{\varphi}_{r,b} (\hat{V}^T \hat{V})_b{}^a = \sum_{s\in A_{g-1}} \varphi_{r,s} \hat{V}_s{}^a \tag{12}$$

where $\hat{V}$ is the fundamental adjacency matrix intertwiner defined in section 1. By equality in (12) we mean that the operators on either side satisfy the same algebra under fusion.

We define a conjugation operator $C(a) = a^*$ to be the identity except for D_{4n} graphs where the eigenvectors Ψ_{am} are complex and conjugation corresponds to the $\mathbb{Z}_{\not\leq}$ Dynkin diagram automorphism. It then follows that $\hat{N}_{a^*b}{}^c = \hat{N}_{ca}{}^b$. We conjecture that the coefficients of the cylinder partition functions $Z_{i_1|i_2}$ of the $sl(2)$ minimal theories are given by the fusion product $\hat{\varphi}_{i_1}^\dagger \times \hat{\varphi}_{i_2}$, that is

$$Z_{i_1|i_2}(q) = \sum_{i_3\in(A,G)} \mathcal{N}_{i_1^* i_2}{}^{i_3} \hat{\chi}_{i_3}(q). \tag{13}$$

More explicitly,

$$Z_{(r_1,a_1)|(r_2,a_2)}(q) = \sum_{(r_3,a_3)\in(A_{h-1},G)} \mathcal{N}_{(r_1,a_1^*)(r_2,a_2)}{}^{(r_3,a_3)} \ \hat{\chi}_{r_3,a_3}(q) \tag{14}$$

$$= \sum_{(r,s)\in(A_{h-1},A_{g-1})} \chi_{r,s}(q) N_{rr_1}{}^{r_2} V_{sa_1}{}^{a_2} \tag{15}$$

where, in terms of Virasoro characters,

$$\hat{\chi}_{r,a}(q) = \sum_{s\in A_{g-1}} \chi_{r,s}(q) \hat{V}_s{}^a. \tag{16}$$

The equivalence of the two forms (14) and (15) of the cylinder partition functions follows from the identity (9) with $a = 1$. The result (15) is not entirely new but generalizes and encompasses several previous results [8, 1, 9]. Note that the matrices $N_r \otimes V_s$ form a representation of the fusion algebra of the minimal model. Indeed, a proper derivation [10] of the above results uses boundary and Ishibashi states and proceeds by showing that Cardy's equation is equivalent to the statement that these coefficient matrices form a representation of the fusion algebra. The problem is thus reduced to finding non-negative integer matrix representations of this algebra.

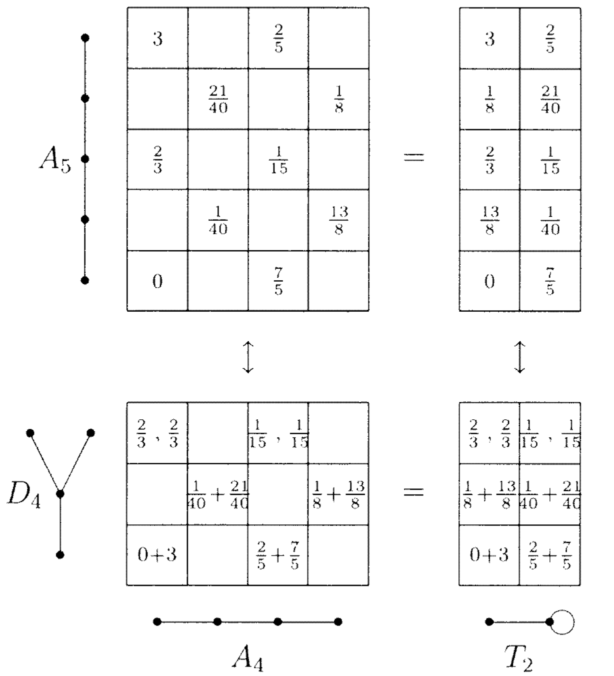

Figure 1. Folding and orbifold duality relating the tensor product graph $T_2 \otimes D_4$ to $A_4 \otimes D_4$ and $A_4 \otimes A_5$. The conformal weights of the eight conformal boundary conditions of the three-state Potts model appear in the boxes of the $T_2 \otimes D_4$ theory.

3. Critical three-state Potts

As an example we consider the $\mathcal{M}(A_4, D_4)$ or critical three-state Potts model. To avoid redundancy, we consider the folded (T_2, D_4) model as shown graphically in figure 1. The tensor product fusion graph $T_2 \otimes D_4$ also arose in the work of Böckenhauer and Evans [11].

The complete list [3, 4] of conformal boundary conditions, conjugate fields $\hat{\varphi}$ and associated characters $\hat{\chi}$ is

$$
\begin{array}{rcllcll}
A & = & (1,1) = (4,1) & \hat{\varphi}_{1,1} & = & I & \chi_0 + \chi_3 \\
B & = & (1,3) = (4,3) & \hat{\varphi}_{1,3} & = & \psi & \chi_{2/3} \\
C & = & (1,4) = (4,4) & \hat{\varphi}_{1,4} & = & \psi^{\dagger} & \chi_{2/3} \\
BC & = & (2,1) = (3,1) & \hat{\varphi}_{2,1} & = & \epsilon & \chi_{2/5} + \chi_{7/5} \\
AC & = & (2,3) = (3,3) & \hat{\varphi}_{2,3} & = & \sigma & \chi_{1/15} \\
AB & = & (2,4) = (3,4) & \hat{\varphi}_{2,4} & = & \sigma^{\dagger} & \chi_{1/15} \\
F & = & (1,2) = (4,2) & \hat{\varphi}_{1,2} & = & \eta & \chi_{1/8} + \chi_{13/8} \\
N & = & (2,2) = (3,2) & \hat{\varphi}_{2,2} & = & \xi & \chi_{1/40} + \chi_{21/40}.
\end{array}
$$

The fused adjacency matrices of $G = D_4$ are

$$V_1 = V_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad V_2 = V_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$V_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{17}$$

The unitary matrix which diagonalizes D_4 is

$$\Psi = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 1 \\ \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \omega & \omega^2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \omega^2 & \omega \end{pmatrix} \tag{18}$$

where $\omega = \exp(2\pi \mathrm{i}/3)$ is a primitive cube root of unity. The graph fusion matrices of D_4 are

$$\hat{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \hat{N}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\hat{N}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{N}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{19}$$

The graph fusion matrices of T_2 are

$$N_1 = N^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad N_2 = N^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{20}$$

The intertwiner $\hat{V}$ and conjugation C are

$$\hat{V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{21}$$

The conjugation operator C acts on the left to raise and lower indices in the fusion matrices $\hat{N}^a = C\hat{N}_a$.

The complete fusion rules of boundary fields are given as follows:

$$\begin{pmatrix} I & \epsilon & \eta & \xi & \psi & \sigma & \psi^\dagger & \sigma^\dagger \\ \epsilon & \epsilon^2 & \epsilon\eta & \epsilon\xi & \epsilon\psi & \epsilon\sigma & \epsilon\psi^\dagger & \epsilon\sigma^\dagger \\ \eta & \eta\epsilon & \eta^2 & \eta\xi & \eta\psi & \eta\sigma & \eta\psi^\dagger & \eta\sigma^\dagger \\ \xi & \xi\epsilon & \xi\eta & \xi^2 & \xi\psi & \xi\sigma & \xi\psi^\dagger & \xi\sigma^\dagger \\ \psi & \psi\epsilon & \psi\eta & \psi\xi & \psi^2 & \psi\sigma & \psi\psi^\dagger & \psi\sigma^\dagger \\ \sigma & \sigma\epsilon & \sigma\eta & \sigma\xi & \sigma\psi & \sigma^2 & \sigma\psi^\dagger & \sigma\sigma^\dagger \\ \psi^\dagger & \psi^\dagger\epsilon & \psi^\dagger\eta & \psi^\dagger\xi & \psi^\dagger\psi & \psi^\dagger\sigma & \psi^{\dagger 2} & \psi^\dagger\sigma^\dagger \\ \sigma^\dagger & \sigma^\dagger\epsilon & \sigma^\dagger\eta & \sigma^\dagger\xi & \sigma^\dagger\psi & \sigma^\dagger\sigma & \sigma^\dagger\psi^\dagger & \sigma^{\dagger 2} \end{pmatrix} = \sum_{r=1}^{2}\sum_{a=1}^{4} N^r \otimes \hat{N}^a \hat{\varphi}_{r,a}$$

$$
= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix} I + \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0 \end{pmatrix} \epsilon
$$

$$
+ \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&1&0&0&0&1&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix} \eta + \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&1&0&0&0&1&0&1\\ 1&1&0&0&1&1&1&1\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0 \end{pmatrix} \xi
$$

$$
+ \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \psi + \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix} \sigma
$$

$$
+ \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix} \psi^\dagger + \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0 \end{pmatrix} \sigma^\dagger
$$

In total, we find 12 distinct cylinder partition functions [1, 3]

$$
\begin{aligned}
&Z_{A|A}(q) = \hat{\chi}_{1,1}(q) = \chi_{1,1}(q) + \chi_{1,5}(q)\\
&Z_{A|B}(q) = \hat{\chi}_{1,4}(q) = \chi_{1,3}(q)\\
&Z_{A|AB}(q) = \hat{\chi}_{2,4}(q) = \chi_{3,3}(q)\\
&Z_{A|BC}(q) = \hat{\chi}_{2,1}(q) = \chi_{3,5}(q) + \chi_{3,1}(q)\\
&Z_{A|F}(q) = \hat{\chi}_{1,2}(q) = \chi_{4,2}(q) + \chi_{4,4}(q)\\
&Z_{A|N}(q) = \hat{\chi}_{2,2}(q) = \chi_{2,2}(q) + \chi_{2,4}(q) = Z_{AB|F}(q)\\
&Z_{AB|AB}(q) = \hat{\chi}_{1,1}(q) + \hat{\chi}_{2,1}(q) = \chi_{1,1}(q) + \chi_{3,5}(q) + \chi_{3,1}(q) + \chi_{1,5}(q)\\
&Z_{AB|AC}(q) = \hat{\chi}_{1,4}(q) + \hat{\chi}_{2,3}(q) = \chi_{3,3}(q) + \chi_{1,3}(q)\\
&Z_{AB|N}(q) = \hat{\chi}_{2,2}(q) + \hat{\chi}_{1,2}(q) = \chi_{2,2}(q) + \chi_{2,4}(q) + \chi_{4,2}(q) + \chi_{4,4}(q)\\
&Z_{F|F}(q) = \hat{\chi}_{1,1}(q) + \hat{\chi}_{1,3}(q) + \hat{\chi}_{1,4}(q) = \chi_{1,1}(q) + \chi_{1,5}(q) + 2\chi_{1,3}(q)\\
&Z_{F|N}(q) = \hat{\chi}_{2,1}(q) + \hat{\chi}_{2,3}(q) + \hat{\chi}_{2,4}(q) = \chi_{3,5}(q) + \chi_{3,1}(q) + 2\chi_{3,3}(q)\\
&Z_{N|N}(q) = \hat{\chi}_{1,1}(q) + \hat{\chi}_{2,1}(q) + \hat{\chi}_{1,3}(q) + \hat{\chi}_{1,4}(q) + \hat{\chi}_{2,3}(q) + \hat{\chi}_{2,4}(q)
\end{aligned}
$$

$$= \chi_{1,1}(q) + \chi_{3,5}(q) + \chi_{3,1}(q) + \chi_{1,5}(q) + 2\chi_{3,3}(q) + 2\chi_{1,3}(q).$$

Here we restrict to Virasoro characters with $r + s$ even. The symmetry

$$Z_{(r_1,a_1)|(r_2,a_2)}(q) = Z_{(r_2,a_2)|(r_1,a_1)}(q) \tag{22}$$

follows because the characters do not distinguish between a field $\hat{\varphi}$ and its conjugate $\hat{\varphi}^{\dagger}$.

4. Integrable boundary weights

We conjecture that all the conformal boundary conditions for $sl(2)$ models can be obtained in the continuum scaling limit of suitably specialized integrable boundary conditions for the associated critical lattice models [12]. For the (A_{g-1}, A_g) theories the integrable boundary weights have been obtained [13], as solutions to the boundary Yang–Baxter equation, by a fusion construction. This method generalizes [13] to the A-D-E models using the appropriate fusion process [14]. The solutions to the boundary Yang–Baxter equation are naturally labelled by a pair (r, a) and are constructed by starting at a and fusing $r - 1$ times. For (A_4, D_4), the non-zero triangular boundary weights attached to the edges of double-row transfer matrices are given explicitly by

$$A, B, C = (1, a): \; 2\Big\langle\begin{smallmatrix}a\\ \vdots\\ a\end{smallmatrix} = 1 \qquad a = 1, 3, 4$$

$$F = (1, 2): \; 1\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 3\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 4\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 1$$

$$BC = (2, 1): \; 3\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 4\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(u), \; 1\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(-u)$$

$$AC = (2, 3): \; 1\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 4\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(u), \; 3\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(-u)$$

$$AB = (2, 4): \; 1\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = 3\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(u), \; 4\Big\langle\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix} = \rho_1(-u)$$

$$N = (2, 2): \; \begin{cases} 2\Big\langle\begin{smallmatrix}b\\ \vdots\\ a\end{smallmatrix} = \rho_2(u) & a \neq b, \quad a, b = 1, 3, 4 \\ 2\Big\langle\begin{smallmatrix}a\\ \vdots\\ a\end{smallmatrix} = \rho_3(u) & a = 1, 3, 4 \end{cases}$$

with u the spectral parameter, $\lambda = \pi/6$, ξ arbitrary and

$$\rho_1(u) = \frac{\sin(u-\lambda-\xi)\sin(u-\lambda+\xi)}{\sin^2\lambda} \qquad \rho_2(u) = \frac{\sin 2u}{\sin 2\lambda}$$

$$\rho_3(u) = \frac{2\sin(u-\xi)\sin(u+\xi)+\sin(u-2\lambda-\xi)\sin(u-2\lambda+\xi)}{\sin^2 2\lambda}.$$

The new boundary condition [3] N is found to be antiferromagnetic in nature. The value of u should be set to its isotropic value $u = \lambda/2$ and ξ chosen appropriately to obtain the conformal boundary conditions.

5. Conclusion

In conclusion we have proposed a set of conjectures that extend the theory of conformal boundaries in a consistent way. The structure of the partition functions is dictated by a new fusion algebra. We comment that the conjecture (15) is independent of the choice of endpoint node and eigenvectors and is meaningful for D_{2n+1} and E_7, even though a proper understanding of the fusion matrices in (14) is missing. We expect the extension to higher rank [15] to be straightforward. A much more comprehensive version of this work will be published elsewhere.

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